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On Eigenfunctions with Sign Definite Components in Weakly Coupled Linear Elliptic Systems

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1. INTRODUCTION

One of the fundamental results in the classical Sturm–Liouville theory of eigenvalue problems for a single second-order ordinary differential operator is that the eigenfunction corresponding to the nth eigenvalue under zero boundary conditions has $n-1$ interior zeros. For elliptic partial differential operators the general question of classifying eigenfunctions remains open, but it has been known for a long time that in the case of a self-adjoint second-order operator with Dirichlet boundary conditions the eigenfunction corresponding to the first eigenvalue does not change sign; see [6]. In the last few years the existence of a real first eigenvalue characterized by a sign-definite eigenfunction has been shown for non-self-adjoint second-order elliptic operators [11] and certain systems of such operators [1, 3] via the theory of positive operators; in the case of systems, positivity is interpreted componentwise, so all components of the first eigenfunction are of one sign. More recently such results have been extended to operators [8] and systems [5, 7] with indefinite weight functions or matrices.

In the present article we consider in more detail the system

$$\begin{aligned} Lu &= \lambda(m_{11}u + m_{12}v) \\ Lv &= \lambda(m_{21}u + m_{22}v) \end{aligned} \quad \text{in } \Omega \tag{1.1}$$

$$u = v = 0 \qquad \text{on } \partial\Omega,$$

where $\Omega \subseteq R^n$ is a bounded domain with smooth boundary, L is a second-order uniformly strongly elliptic operator of the form

$$Lw = -\sum_{i,j=1}^{n} a_{ij}(x)\, w_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\, w_{x_i} + c(x)\, w \tag{1.2}$$

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with $a_{ij} = a_{ji}$, $c \geqslant 0$, and coefficients in $C^\alpha(\bar{\Omega})$ for some $\alpha \in (0, 1)$, and $m_{ij}(x) > 0$ on $\bar{\Omega}$, $m_{ij} \in C^\alpha(\bar{\Omega})$ for $i, j = 1, 2$. We shall sometimes use M to denote the matrix (m_{ij}). We study two types of solutions to (1.1). The first type of solution is the type shown to exist in [1], with both components positive. For such solutions we consider a problem that only makes sense in the context of systems, namely that of determining the relative size (alternatively, bounding the ratio) of the components in terms of the matrix M. The second type consists of those solutions (if any) with both components nonzero in Ω but of opposite signs. Such solutions were observed to exist in the case where M is a constant matrix satisfying appropriate conditions in [2]; we consider the question of existence of such solutions in the general case where M may vary with x. Our results are not complete, but they indicate that the existence of such solutions depends on the structure of the system (1.1) in a very delicate way, and make it appear unlikely that existence results for such solutions can be obtained by a straightforward application of the theory of positive operators on cones.

Some of our ideas and results are related to the concept of generalized spectrum for (1.1). The generalized spectrum consists of the set of pairs (λ, μ) for which the problem

$$\begin{aligned} Lu &= \lambda(m_{11}u + m_{12}v) \\ Lv &= \mu(m_{21}u + m_{22}v) \end{aligned} \quad \text{in } \Omega \qquad u = v = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

has a nontrivial solution (u, v). The term generalized spectrum was introduced by Protter in [10] in connection with estimating eigenvalues; similar ideas arise naturally in multiparameter bifurcation/continuation problems; see [2]. Properties of the generalized spectrum are discussed in [3]. It follows from results of [1, 3] that for any fixed positive value of the ratio μ/λ, (1.3) will have a solution with both u and v positive in Ω for some λ. In the present article we address the question of how the ratio u/v behaves as λ/μ varies, especially when λ/μ tends to zero or infinity.

In our analysis in [4] of the stability of steady states of a reaction diffusion model from mathematical biology, we found that some relevant bounds for eigenvalues of the linearized system depended on the relative sizes of the components of an eigenvector. (This is not explicitly stated in [4] but is implicit in some of the computations.) The present article does not fully answer the questions raised by [4], but gives a first step toward an answer.

Remarks on Notation

At various places in what follows we will want to compare functions which are zero on $\partial\Omega$. To be sure that ratios of such functions exist and are finite we need to know something about their behavior on $\partial\Omega$. It follows from the strong maximum principle that if $Lw \geqslant 0$ in Ω, $w \geqslant 0$ in Ω, $w = 0$ on $\partial\Omega$, and $w \not\equiv 0$, then $w > 0$ in Ω and $\partial w/\partial n < 0$ on $\partial\Omega$, where $\partial/\partial n$ denotes the outer normal derivative. We shall write $w \gg 0$ if $w > 0$ in Ω, $w = 0$ on $\partial\Omega$, and $\partial w/\partial n < 0$ on $\partial\Omega$. The above discussion shows that if ϕ is the eigenfunction corresponding to the first eigenvalue $\lambda_1(m)$ for the single equation $L\phi = \lambda m\phi$ in Ω, $\phi = 0$ on $\partial\Omega$, then $\phi \gg 0$. The significance of these observations is that if ϕ, $\psi \gg 0$ then the supremum of ϕ/ψ over $\bar{\Omega}$ is defined and finite.

In looking for solutions to (1.1) with, say, $u > 0$ and $v < 0$, we will sometimes find it convenient to replace v by $w = -v$ and consider the system

$$\begin{aligned} Lu &= \lambda(m_{11}u - m_{12}w) \\ Lv &= \lambda(-m_{21}u + m_{22}w). \end{aligned} \tag{1.4}$$

The question of solving (1.1) with $u > 0$ and $v < 0$ is equivalent to that of solving (1.4) with $u > 0$ and $w > 0$; we shall utilize that equivalence in what follows without further comment.

2. Sum Techniques

The results of this section are based on the following:

Lemma 2.1 (Positivity Lemma). *Suppose that $m(x) \in C^\alpha(\bar{\Omega})$. There is a unique value $\lambda_1(m)$ with $\lambda_1(m) \in (0, \infty)$ if $m(x_0) > 0$ for some $x_0 \in \Omega$ and $\lambda_1(m) = \infty$ if $m \leqslant 0$ on Ω such that the problem*

$$Lu - \lambda m u = f \qquad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega \tag{2.1}$$

with $\lambda \geqslant 0$, $f \geqslant 0$, and $f \not\equiv 0$ has a positive solution if and only if $\lambda < \lambda_1(m)$. If $\lambda < \lambda_1(m)$, the solution of (2.1) is unique. The value $\lambda_1(m)$ is nonincreasing with respect to m.

Discussion. Lemma 2.1 is essentially contained in the results of [8], which are based in large part on the theory of positive operators. If $m(x_0) > 0$ at $x_0 \in \Omega$, then $\lambda_1(m)$ is the eigenvalue with least positive real part for the problem $Lu = \lambda m u$ in Ω, $u = 0$ on $\partial\Omega$; $\lambda_1(m)$ is the only positive eigenvalue with positive eigenfunction.

We now consider the system

$$\begin{aligned} Lu &= \lambda(m_{11}u + m_{12}v) \\ Lv &= \lambda(m_{21}u + m_{22}v) \end{aligned} \quad \text{in } \Omega \qquad u = v = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

where $m_{ij} \in C^\alpha(\bar{\Omega})$; $m_{ij} > 0$ in $\bar{\Omega}$ for $i, j = 1, 2$.

THEOREM 2.2. *If u and v satisfy* (2.2) *and are positive in* Ω, *then*

$$\beta v \leqslant u \quad \text{and} \quad u \leqslant \gamma v, \tag{2.3}$$

where

$$\beta = \sup\{\alpha \geqslant 0 \colon m_{12} + \alpha(m_{11} - m_{22}) - \alpha^2 m_{21} > 0 \text{ in } \bar{\Omega}\}$$

$$\gamma = \inf\{\alpha \geqslant 0 \colon m_{12} + \alpha(m_{11} - m_{22}) - \alpha^2 m_{21} < 0 \text{ in } \bar{\Omega}\}.$$

Proof. First we observe that (2.2) implies that $Lu - \lambda m_{11} u = \lambda m_{12} v > 0$, so by Lemma 2.1 we have $\lambda < \lambda_1(m_{11}) \leqslant \lambda_1(m_{11} - \alpha m_{12})$ for any $\alpha \geqslant 0$. Multiplying the second equation in (2.2) by α and subtracting it from the first yields

$$L(u - \alpha v) - \lambda[m_{11} - \alpha m_{21}](u - \alpha v) = \lambda[m_{12} + \alpha(m_{11} - m_{22}) - \alpha^2 m_{21}]\, v.$$

Since $\lambda < \lambda_1(m_{11} - \alpha m_{21})$, it follows from Lemma 2.1 that if $m_{12} + \alpha(m_{11} - m_{22}) - \alpha^2 m_{21} > 0$ then $u - \alpha v > 0$ or $\alpha v < u$. Taking the supremum over such $\alpha \geqslant 0$ yields the first inequality in (2.3). A similar analysis of the case $m_{12} + \alpha(m_{11} - m_{22}) - \alpha^2 m_{21} < 0$ yields the second inequality in (2.3).

Remarks. Let $\bar{m}_{ij} = \sup\{m_{ij}(x) \colon x \in \bar{\Omega}\}$ and $\underline{m}_{ij} = \inf\{m_{ij}(x) \colon x \in \bar{\Omega}\}$. Then for $\alpha \geqslant 0$,

$$m_{12} + \alpha(m_{11} - m_{22}) - \alpha^2 m_2 \geqslant \underline{m}_{12} + \alpha(\underline{m}_{11} - \underline{m}_{22}) - \alpha^2 \bar{m}_{21}.$$

Thus

$$\beta \geqslant \{\underline{m}_{11} - \bar{m}_{22} + [(\bar{m}_{22} - \underline{m}_{11})^2 + 4\underline{m}_{12}\bar{m}_{21}]^{1/2}\}/2\bar{m}_{21} \tag{2.4}$$

and

$$\gamma \leqslant \{\bar{m}_{11} - \underline{m}_{22} + [(\underline{m}_{22} - \bar{m}_{11})^2 + 4\bar{m}_{12}\underline{m}_{21}]^{1/2}\}/2\underline{m}_{21}. \tag{2.5}$$

We can apply Theorem 2.2 and the remarks following it to generalized spectrum problems as discussed in [3]. Suppose u and v are positive in Ω and satisfy

$$\left.\begin{aligned} Lu &= \lambda(m_{11} u + m_{12} v) \\ Lv &= \mu(m_{21} u + m_{22} v) \end{aligned}\right\} \text{ in } \Omega \qquad u = v = 0 \quad \text{on } \partial\Omega \tag{2.6}$$

with λ, μ positive. The system (2.6) can be rewritten as

$$\begin{aligned} Lu &= \lambda(m_{11} u + m_{12} v) \\ Lv &= \lambda(\sigma m_{21} u + \sigma m_{22} v) \end{aligned} \tag{2.7}$$

with $\sigma = \mu/\lambda$. If we now apply (2.3), (2.4), and (2.5) to system (2.7) we obtain the estimates

$$\{\underline{m}_{11} - \sigma\bar{m}_{22} + [(\sigma\bar{m}_{22} - \underline{m}_{11})^2 + 4\sigma\underline{m}_{12}\bar{m}_{21}]^{1/2}\}/2\bar{m}_{21}\sigma \leqslant u/v \tag{2.8}$$

and

$$u/v \leqslant \{\bar{m}_{11} - \sigma\underline{m}_{22} + [(\sigma\underline{m}_{22} - \bar{m}_{11})^2 + 4\sigma\bar{m}_{12}\underline{m}_{21}]^{1/2}\}/2\underline{m}_{21}\sigma.$$

It follows from [3] that (2.6) has positive solutons u and v for (λ, μ) lying on an arc in the first quadrant of the $\lambda - \mu$ plane which connects points $(\lambda_0, 0)$ and $(0, \mu_0)$, where λ_0, u_0 and μ_0, v_0 are the first eigenvalues and eigenfunctions for the problems $Lu = \lambda m_{11} u$ in Ω, $u = 0$ on $\partial\Omega$ and $Lv = \lambda m_{22} v$ in Ω, $v = 0$ on $\partial\Omega$, respectively. As $(\lambda, \mu) \to (\lambda_0, 0)$ along the arc, the normalized eigenvector $(u, v) \to (u_0, 0)$; similarly as $(\lambda, \mu) \to (0, \mu_0)$, $(u, v) \to (0, v_0)$. The estimates (2.8) give bounds on the rate that u/v blows up as $(\lambda, \mu) \to (\lambda_0, 0)$ or goes to zero as $(\lambda, \mu) \to (0, \mu_0)$. An examination of (2.8) shows that for positive u, v satisfying (2.6) we have $u/v \sim \lambda/\mu$ as $(\lambda/\mu) \to 0$ or $(\lambda/\mu) \to \infty$.

Next we give a result providing a necessary condition for the existence of a solution to (2.2) with one component positive and one negative.

THEOREM 2.3. *Suppose that* $\sup m_{11}(x) \sup m_{22}(x) - \inf m_{12}(x) \inf m_{21}(x) < 0$. *Then for* $\lambda \geqslant 0$, (2.2) *cannot have a solution* u, v *with* $u > 0$ *and* $v < 0$ *in* Ω, *or* $u < 0$ *and* $v > 0$ *in* Ω.

Proof. Since $\sup m_{11}(x) \sup m_{22}(x) - \inf m_{12}(x) \inf m_{21}(x) < 0$, we have $\sup(m_{11}/m_{21}) \leqslant \sup m_{11}/\inf m_{21} < \inf m_{12}/\sup m_{22} \leqslant \inf(m_{12}/m_{22})$; thus there exists an α such that $0 < m_{11}/m_{21} < \alpha < m_{12}/m_{22}$ in $\bar{\Omega}$, that is, $m_{12} - \alpha m_{22} > 0$ and $\alpha m_{21} - m_{11} > 0$. Suppose that $u > 0$ and $v < 0$ satisfy (2.2). (If $u < 0$, $v > 0$, consider $(-u, -v)$.) Multiplying the second equation in (2.2) by α and subtracting the first from it, we obtain

$$\begin{aligned} L(\alpha v - u) &= \lambda[\alpha m_{21} u + \alpha m_{22} v - m_{11} u - m_{12} v] \\ &= \lambda[(\alpha m_{21} - m_{11})\, u - (m_{12} - \alpha m_{22})\, v] > 0. \end{aligned}$$

Thus, by Lemma 2.1, $\alpha v - u > 0$. However, $u > 0$ and $v < 0$, so $\alpha v - u < 0$, which yields a contradiction. Thus, there can be no solution with sign definite components of opposite signs.

3. SIMILARITY METHODS

The results of the last section were based on the observation that for the type of systems we consider linear combinations of the components of a solution satisfy equations related to those occurring in the system. We now examine the effect of applying certain linear transformations to the system as a whole, and show how systems whose eigenvectors have certain desired properties (such as sign definite components) can be generated. Let $\mathscr{L}=LI$, where I is the 2×2 identity matrix, let $\mathbf{w}=\operatorname{col}(u, v)$, and let $M=(m_{ij})$. We can write our system as

$$\begin{aligned} \mathscr{L}\mathbf{w} &= \lambda M\mathbf{w} && \text{in } \Omega \\ \mathbf{w} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Our basic observation is that if S is a nonsingular 2×2 matrix of constants then S and S^{-1} commute with $\mathscr{L}$; if we let $\mathbf{z}=S^{-1}\mathbf{w}$ then we have

$$\begin{aligned} \mathscr{L}\mathbf{z} &= S^{-1}\mathscr{L}S\mathbf{z}=S^{-1}\mathscr{L}\mathbf{w}=\lambda S^{-1}MS\mathbf{z} && \text{in } \Omega, \\ \mathbf{z} &= S^{-1}\mathbf{w}=0 && \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Thus, we can allow the group of automorphisms generated by the invertible 2×2 constant matrices to act on M, and we see that the spectrum of (3.1) is preserved under that action. Also, if we have a system which admits a solution $\mathbf{w}$ with some specified property such as sign definiteness of its components, we can construct other systems with solutions possessing that property by finding matrices S such that S^{-1} preserves the property in question and then using (3.2).

For applications to bifurcation theory it is important to view λ as a characteristic value of $\mathscr{L}^{-1}M$ and to decide if its algebraic and geometric multiplicities are the same. Such will be the case provided the nullspaces for $I-\lambda\mathscr{L}^{-1}M$ and $(I-\lambda\mathscr{L}^{-1}M)^2$ are the same. It turns out that this property is preserved under similarity transformations, as can be seen by a routine calculation. We have

PROPOSITION 3.1. *Suppose that* $[I-\lambda\mathscr{L}^{-1}M]^2\mathbf{v}=0$ *implies* $[I-\lambda\mathscr{L}^{-1}M]\mathbf{v}=0$, *provided* $\mathbf{v}\in[C_0^\alpha(\Omega)]^2$. *Then for any nonsingular* 2×2 *constant matrix* S *and* $\mathbf{v}\in[C_0^\alpha(\Omega)]^2$, $[I-\lambda\mathscr{L}^{-1}S^{-1}MS]^2\mathbf{v}=0$ *implies* $[I-\lambda\mathscr{L}^{-1}S^{-1}MS]\,\mathbf{v}=0$.

Suppose now that

$$M=\begin{pmatrix} m_{11} & -m_{12} \\ -m_{21} & m_{22} \end{pmatrix}$$

and that system (3.1) has a solution $\mathbf{w}=\operatorname{col}(u, v)$ with $u\gg 0$ and $v\gg 0$. Then the system with M replaced by $S^{-1}MS$ has a solution $\mathbf{z}=S^{-1}\mathbf{w}$.

Moreover $\mathbf{z}$ has strictly positive components provided that S^{-1} is a positive definite constant matrix with nonnegative entries. In particular, we have the following hyperbolicity principle.

THEOREM 3.2. *Suppose that* (3.1) *has a solution* $\mathbf{w}=\operatorname{col}(u, v)$ *with* $u\gg 0$ *and* $v\gg 0$, *where*

$$M=\begin{pmatrix} m_{11} & -m_{12} \\ -m_{21} & m_{22} \end{pmatrix}$$

and $m_{ij}>0$ *on* $\bar{\Omega}$. *Then if*

$$M_\alpha=\begin{pmatrix} m_{11} & -(1/\alpha)\,m_{12} \\ -\alpha m_{21} & m_{22} \end{pmatrix}$$

the system $\mathscr{L}\mathbf{z}=\lambda M_\alpha\mathbf{z}$ *has a solution with positive components for any* $\alpha>0$.

Proof.

$$M_\alpha=\begin{pmatrix} (1/\alpha) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m_{11} & -m_{12} \\ -m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us now consider the following special system:

$$\begin{aligned} Lx &= \lambda[m_{11}x-m_{12}y] \\ Ly &= \lambda[-m_{12}x+m_{11}y] \end{aligned} \quad \text{in } \Omega \tag{3.3}$$
$$x\equiv 0\equiv y \quad \text{on } \partial\Omega,$$

where $m_{11}, m_{12}>0$ on $\bar{\Omega}$ and $(m_{11}-m_{12})(x_0)>0$ for some $x_0\in\Omega$. Observe that if (3.3) is satisfied then

$$L(x+y)=\lambda(m_{11}-m_{12})(x+y)$$

and

$$L(x-y)=\lambda(m_{11}+m_{12})(x-y).$$

Consequently, $x+y=\alpha z$, where $Lz=\lambda(m_{11}-m_{12})\,z$, and $x-y=\beta w$, where $Lw=\lambda(m_{11}+m_{12})\,w$. Hence if $\lambda=\lambda_1(m_{11}-m_{12})>\lambda_1(m_{11}+m_{12})$, system (3.3) has a componentwise positive solution. Moreover, $\lambda_1(m_{11}-m_{12})$ will be a simple eigenvalue for (3.3) provided that $\lambda_1(m_{11}-m_{12})$ is not an eigenvalue of

$$\begin{aligned} Lx &= \sigma(m_{11}+m_{22})\,w && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

(Consequently, eigenvalues for (3.1) admitting componentwise positive solutions need not be simple, which, as noted in Section 1, suggests that the Krein–Rutman theorem is not well-suited to the investigation of the existence of componentwise positive solutions to (3.1).)

We may now use (3.3) and similarity transformations to generate a class of matrices M for which (3.1) has a componentwise positive solution.

THEOREM 3.3. *Suppose that* $h, k \in C^{\alpha}(\bar{\Omega})$ *with* $h > k > 0$ *on* $\bar{\Omega}$. *Let* $\sigma > 0$ *be such that* $\sigma < \max_{x \in \bar{\Omega}} (h(x)/k(x))$. *Then*

$$M = \begin{pmatrix} h+k & -k \\ -(\sigma^2-1)\,k & h-k \end{pmatrix}$$

is such that $Lw = \lambda M w$ *admits a componentwise positive solution for some* $\lambda > 0$. *Moreover, if* $\sigma > 1$, M *has the form*

$$\begin{pmatrix} m_{11} & -m_{12} \\ -m_{21} & m_{22} \end{pmatrix}$$

with $m_{ij} > 0$.

Proof. Consider the system

$$\begin{aligned} Lu &= \lambda[hu - \sigma k v] \\ Lv &= \lambda[-\sigma k u + h v]. \end{aligned} \tag{3.4}$$

Since $(h - \sigma k)(x_0) >$ for some x_0, (3.4) is of the form (3.3) and admits a componentwise positive solution when $\lambda = \lambda_1(h - \sigma k)$. Let

$$S = \begin{pmatrix} 1 & 0 \\ -1/\sigma & 1/\sigma \end{pmatrix}.$$

Then

$$S^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & \sigma \end{pmatrix} \quad \text{and} \quad M = S^{-1} \begin{pmatrix} h & -\sigma k \\ -\sigma k & h \end{pmatrix} S$$

and the result follows from the discussion preceding Theorem 3.2.

COROLLARY 3.4. *Suppose that* $f, g \in C^{\alpha}(\bar{\Omega})$ *with* $f > g > 0$ *on* $\bar{\Omega}$. *Then if*

$$M = \begin{pmatrix} f & -\sigma(f-g) \\ -\delta(f-g) & g \end{pmatrix} \quad \textit{and} \quad \gamma = \sigma\delta < \max_{x \in \bar{\Omega}} \left(\frac{(fg)(x)}{(f-g)^2(x)} \right),$$

there is a $\lambda > 0$ *so that* (3.1) *admits a componentwise positive solution.*

Proof. Theorems 3.2 and 3.3 imply that it suffices to show that

$$\begin{pmatrix} f & -(f-g)/2 \\ -4\gamma((f-g)/2) & g \end{pmatrix}$$

has the form

$$\begin{pmatrix} h+k & -k \\ -(\sigma^2-1)\,k & h-k \end{pmatrix}$$

with $h, k \in C^{\alpha}(\bar{\Omega})$, $h > k > 0$, and $1 < \sigma < \max_{x \in \bar{\Omega}} (h(x)/k(x))$. Taking $h = (f+g)/2$ and $k = (f-g)/2$, we need only show $\sqrt{4\gamma + 1} < \max_{x \in \bar{\Omega}} (h(x)/k(x))$. Now $\gamma < \max_{x \in \bar{\Omega}} ((fg)(x)/(f-g)^2(x)) = \max_{x \in \bar{\Omega}} ((h^2 - k^2)(x)/(4k^2)(x))$. Hence $4\gamma < \max_{x \in \bar{\Omega}} [h^2(x)/k^2(x) - 1] = [\max_{x \in \bar{\Omega}} (h^2(x)/k^2(x))] - 1$. Thus $4\gamma + 1 < \max_{x \in \bar{\Omega}} [h(x)/k(x)]^2$ and so $\sqrt{4\gamma + 1} < \max_{x \in \bar{\Omega}} (h(x)/k(x))$.

Remark. From Theorem 2.3, a necessary condition for the existence of a componentwise positive solution is

$$\sigma\delta < \frac{(\sup f)(\sup g)}{[\inf(f-g)]^2}.$$

Consider once again (3.1) in the special case where

$$M = \begin{pmatrix} f & -\sigma(f-g) \\ -\delta(f-g) & g \end{pmatrix}$$

and f, g, σ, and δ are as in the statement of Corollary 3.4. By Corollary 3.4, there are solutions to this system with positive components of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where a, b, c, d are nonnegative, $ad - bc \neq 0$, and $\mathbf{w} = \binom{u}{v}$ is a componentwise positive solution to $\mathscr{L}\mathbf{w} = \lambda \tilde{M} \mathbf{w}$ with $\tilde{M}$ of the form

$$\begin{pmatrix} m_{11} & -m_{12} \\ -m_{12} & m_{11} \end{pmatrix}.$$

From the discussion following system (3.4) we may take $u = v$. Consequently

$$\mathscr{L}\mathbf{w} = \lambda M \mathbf{w} \quad \left(M = \begin{pmatrix} f & -\sigma(f-g) \\ -\delta(f-g) & g \end{pmatrix} \right)$$

has componentwise positive solutions for some $\lambda > 0$ with components which are multiples of each other. We conclude this section by observing a slightly more general condition on M which must be satisfied in order that $\mathscr{L}\mathbf{w} = \lambda M \mathbf{w}$ have solutions $\mathbf{w} = \binom{\alpha v}{v}$ with $v \gg 0$ and $\alpha > 0$. Namely, suppose that $\mathbf{w}$ is a solution; then by eliminating Lv from the resulting system and simplifying we find that we must have $m_{21}\alpha^2 + (m_{11} - m_{22})\alpha - m_{12} \equiv 0$.

4. Perturbation Techniques

Let us now consider

$$\begin{aligned} Lu &= \lambda(m_{11}u - tm_{12}v) \\ Lv &= \lambda(-tm_{21}u + m_{22}v) \end{aligned} \quad \text{in } \Omega, \tag{4.1}$$

where

$$u \equiv 0 \equiv v \quad \text{on } \Omega \tag{4.2}$$

and t is a sufficiently small positive number. In addition, throughout this section we assume L to be formally self-adjoint. It is evident that (4.1)–(4.2) satisfies the necessary conditions of Theorem 2.3 for the existence of a componentwise positive solution so long as t is sufficiently small under the assumption of positivity on m_{ij}, $i, j = 1, 2$. We now explore the question of the existence of componentwise positive solutions to (4.1)–(4.2) in closer detail. Let $\lambda_1(m_{11})$ and $\lambda_1(m_{22})$ denote the unique positive eigenvalues of

$$\begin{aligned} Lw &= \lambda m_{ii} w && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega \end{aligned}$$

for $i = 1, 2$, respectively, admitting positive eigenfunctions ϕ and ψ, respectively.

Lemma 4.1. *Let $\varepsilon > 0$ be given. Then there is a $t_0 > 0$ such that if* (4.1)–(4.2) *admits a positive solution for some (λ, t) with $0 < t < t_0$ and $\lambda > 0$, then* $\max(\lambda_1(m_{11}), \lambda_1(m_{22})) < \lambda < (1+\varepsilon)\max(\lambda_1(m_{11}), \lambda_1(m_{22}))$.

Proof. Suppose that $u \gg 0$ and $v \gg 0$ solve (4.1)–(4.2) for some $\lambda > 0$ and $t > 0$. Consider the first equation of (4.1). Multiplying both sides of the equation by ϕ and integrating by parts via the divergence theorem yields

$$\lambda_1(m_{11}) \int_\Omega m_{11}\phi u = \lambda \int_\Omega m_{11}\phi u - \lambda t \int_\Omega m_{12}\phi v.$$

Since $\int_\Omega m_{11}\phi u$ and $\int_\Omega m_{12}\phi v$ are positive, $\lambda_1(m_{11}) < \lambda$. A similar argument gives $\lambda_1(m_{22}) < \lambda$. So $\max(\lambda_1(m_{11}, \lambda_1(m_{22})) < \lambda$.

Once again, consider the equations

$$\lambda_1(m_{11}) \int_\Omega m_{11}\phi u = \lambda \int_\Omega m_{11}\phi u - \lambda \int_\Omega tm_{12}\phi v$$

and

$$\lambda_1(m_{22}) \int_\Omega m_{22}\psi v = -\lambda \int_\Omega tm_{21}\psi u + \lambda \int_\Omega m_{22}\psi v$$

obtained as described above. Combining the equations we have

$$\lambda \int (m_{11}\phi - tm_{21}\psi)\, u + \lambda \int_\Omega (m_{22}\psi - tm_{12}\phi) v$$

$$= \lambda_1(m_{11}) \int_\Omega m_{11}\phi u + \lambda_1(m_{22}) \int_\Omega m_{22}\psi v.$$

But now there exists a $t_0 > 0$ such that $0 < t < t_0$ implies

$$m_{11}\phi - tm_{21}\psi > \frac{1}{1+\varepsilon} m_{11}\phi$$

on Ω and that

$$m_{22}\psi - tm_{12}\phi > \frac{1}{1+\varepsilon} m_{22}\psi$$

on Ω. (That such is the case follows since $\phi, \psi \gg 0$.) Consequently, if $0 < t < t_0$,

$$\frac{\lambda}{1+\varepsilon} \int_\Omega m_{11}\phi u + \frac{\lambda}{1+\varepsilon} \int_\Omega m_{22}\psi v < \lambda_1(m_{11}) \int_\Omega m_{11}\phi u + \lambda_1(m_{22}) \int_\Omega m_{22}\psi v.$$

Hence $\lambda < (1+\varepsilon)\max\{\lambda_1(m_{11}), \lambda_1(m_{22})\}$.

Let $\bar{\lambda} = \max\{\lambda_1(m_{11}), \lambda_1(m_{22})\}$. Let $\| \ \|_\alpha$ denote the norm of the Hölder space $C^\alpha(\bar{\Omega})$, and recall that by standard elliptic theory L^{-1} is compact on $C^\alpha(\bar{\Omega})$. If there exists a sequence $\{(t_n, \lambda_n, u_n, v_n)\}_{n=1}^\infty$ of solutions to (4.1)–(4.2) with $t_n \to 0$, $\lambda_n > 0$, $u_n \gg 0$, $v_n \gg 0$, and $\|u_n\|_\alpha + \|v_n\|_\alpha = 1$, then

$\lambda_n \to \bar{\lambda}$ and by compactness of L^{-1} we may assume $u_n \to \bar{u}$, $v_n \to \bar{v}$, where $\bar{u} \geqslant 0$, $\bar{v} \geqslant 0$, $\|\bar{u}\|_\alpha + \|\bar{v}\|_\alpha = 1$, and

$$\begin{aligned} L\bar{u} &= \bar{\lambda} m_{11} \bar{u} \\ L\bar{v} &= \bar{\lambda} m_{22} \bar{v}. \end{aligned} \quad \text{in } \Omega \tag{4.3}$$

$$\bar{u} \equiv 0 \equiv \bar{v} \quad \text{on } \partial\Omega. \tag{4.4}$$

Hence componentwise positive solutions to (4.1)–(4.2) only arise as perturbations of nonnegative solutions to (4.3)–(4.4). Since L^{-1} is compact, (4.1)–(4.2) may be recast as an eigenvalue problem for a compact linear operator $A(t)$ depending analytically on t. If $\lambda_1(m_{11}) \neq \lambda_1(m_{22})$, then $1/\bar{\lambda}$ is an algebraically simple eigenvalue for $A(0)$ and the results of [9] guarantee the existence of a smooth curve of eigenvalues and normalized eigenfunctions for $A(t)$, $|t|$ small. (See [3] for further discussion.) Let us now suppose that $\lambda_1(m_{11}) < \lambda_1(m_{22})$. (It is easy to see that the assumption $m_{22}(x) < m_{11}(x)$ on Ω is sufficient in this regard.) Consequently, we may assume that a smooth curve of solutions has been selected and formally differentiate (4.1).

So doing, we obtain

$$\begin{aligned} Lu' &= \lambda'(m_{11} u - t m_{12} v) + \lambda(m_{11} u' - t m_{12} v' - m_{12} v) \\ Lv' &= \lambda'(-t m_{21} u + m_{22} v) + \lambda(-t m_{21} u' + m_{22} v' - m_{21} u), \end{aligned} \tag{4.5}$$

where λ', u', v' indicate the derivatives of λ, u, and v with respect to t. If now, $t = 0$, $\lambda = \lambda_1(m_{22})$, $u = 0$, $v = \psi$, (4.5) reduces to

$$\begin{aligned} Lu'(0) &= \lambda_1(m_{22})(m_{11} u'(0) - m_{12} \psi) \\ Lv'(0) &= \lambda'(0)\, m_{22}\, \psi + \lambda_1(m_{22})\, m_{22} v'(0). \end{aligned} \tag{4.6}$$

Consider now (4.6). Since $\psi \gg 0$, $v(t) \gg 0$ for $t > 0$ and small. Consequently, (4.1)–(4.2) has a positive solution for $\lambda > 0$ and $t > 0$ and small if $u'(0) \gg 0$. We have now established the following result.

THEOREM 4.2. *Suppose that $\lambda_1(m_{11})$ and $\lambda_1(m_{22})$ denote the unique positive eigenvalues of*

$$\begin{aligned} Lw &= \lambda m_{ii} w \quad && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega \end{aligned}$$

admitting positive eigenfunctions ϕ and ψ, respectively, $i = 1, 2$. If $\lambda_1(m_{11}) < \lambda_1(m_{22})$ and $(L - \lambda_1(m_{22})\, m_{11})^{-1}$ exists, then (4.1)–(4.2) has componentwise positive solutions with $\lambda > 0$ for $t > 0$ and small if

$$-(L - \lambda_1(m_{22})\, m_{11})^{-1}(m_{12}\, \psi) \gg 0.$$

COROLLARY 4.3. *Suppose now that $m_{11} > m_{22}$ on Ω, $(L - \lambda_1(m_{22})\, m_{11})^{-1}$ exists, and that $m_{12} = k(m_{11} - m_{22})$, where k is a positive constant. Then the result of Theorem 4.2 obtains.*

Proof. Observe that $(L - \lambda_1(m_{22})\, m_{11})\, \psi = (L - \lambda_1(m_{22})\, m_{22})\, \psi + \lambda_1(m_{22})(m_{22} - m_{11})\, \psi = -(\lambda_1(m_{22})/k)\, m_{12} \psi$. Consequently, $(k/\lambda_1(m_{22}))\, \psi = -(L - \lambda_1(m_{22})\, m_{11})^{-1}(m_{12} \psi)$.

Corollary 4.3 has an immediate consequence, which connects Theorem 4.2 to the results of [2].

COROLLARY 4.4. *If m_{ij}, $i, j = 1, 2$, are positive constants, with $m_{11} > m_{22}$ and $(L - \lambda_1(m_{22})\, m_{11})^{-1}$ exists, the result of Theorem 4.2 obtains.*

An additional criterion for the result of Theorem 4.2 to hold may be seen as follows. Since $L\phi = \lambda_1(m_{11})\, m_{11} \phi$, we have that $(L - \lambda_1(m_{22})\, m_{11})\, \phi = (\lambda_1(m_{11}) - \lambda_1(m_{22}))\, m_{11} \phi$, and so

$$\frac{\lambda_1(m_{22})}{\lambda_1(m_{22}) - \lambda_1(m_{11})}\, \phi = -\lambda_1(m_{22})(L - \lambda_1(m_{22})\, m_{11})^{-1}(m_{11} \phi).$$

Consequently, since $(L - \lambda_1(m_{22})\, m_{11})^{-1}$ may be viewed as a continuous operator between the Hölder spaces $C^\alpha(\bar{\Omega})$ and $C_0^{2+\alpha}(\bar{\Omega})$ and $\phi \gg 0$, it follows that $u'(0) \gg 0$ provided $\|m_{11} \phi - m_{12} \psi\|_\alpha$ is sufficiently small. ($\| \; \|_\alpha$ denotes the usual norm in $C^\alpha(\bar{\Omega})$.)

Finally, we give an example of a system of the form (4.1)–(4.2) which has no positive solutions for $t > 0$ and small. Suppose that $\Omega = [0, \Pi]$, $L = -(d^2/dx^2)$, $m_{11} = a \in (1, 4)$, $m_{22} = 1$, and $m_{12} = m_{21} = (\Pi/2)(2 + \cos x)$. Then $\lambda_1(m_{11}) = 1/a$, $\phi = (2/\Pi) \sin x$, $\lambda_1(m_{22}) = 1$, $\psi = (2/\Pi) \sin x$. The top equation of (4.6) then yields

$$\begin{aligned} \frac{-d^2(w_a)}{dx^2} - a w_a &= -2 \sin x - \sin x \cos x \\ w_a(0) = w_a(\Pi) &= 0, \end{aligned} \tag{4.7}$$

where $w_a(x) = [u'(0)]_a(x)$. An elementary calculation will show that

$$\begin{aligned} w_a &= \frac{2}{a-1} \sin x + \frac{1}{2(a-4)} \sin 2x \\ &= \sin x \left(\frac{2}{a-1} + \frac{1}{a-4} \cos x \right). \end{aligned}$$

Hence $w_a(x) \gg 0$ exactly as $h_a(x) = 2/(a-1) + (1/(a-4)) \cos x > 0$ on $[0, \Pi]$. Since $h_a'(x) = (1/4 - a) \sin x > 0$ on $(0, \Pi)$, $h_a(x) > 0$ on $[0, \Pi]$

provided $h_a(0) > 0$. But $h_a(0) = (3(a-3)/(a-1)(a-4))$. Consequently, if $a \in (3,4)$, the system has no positive solution for $t > 0$ and sufficiently small.

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